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A three-dimensional analogue of the Prandtl–Batchelor closed streamline theory

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For steady laminar flow with closed streamlines Batchelor (1956) has shown how an integral condition arising from the effect of viscosity can be used with the inviscid flow equations to determine the vorticity distribution when the Reynolds number is large. Here a condition analogous to that used by Batchelor is derived for a class of flows with helical streamlines. An exact integral condition relating the constant axial pressure gradient and the viscous terms is obtained, which when combined with the inviscid flow equations leads to the result that the axial velocity is proportional to the stream function for the motion in the plane normal to the axial velocity.

1. Introduction

When the equations of steady motion for a viscous incompressible fluid are integrated around a closed streamline we obtain the exact result (Batchelor 1956)

$$\oint \operatorname{curl} \boldsymbol{\omega} . d\mathbf{s} = 0, \tag{1.1}$$

where $\boldsymbol{\omega}$ is the vorticity and $d\mathbf{s}$ is the line element in the streamline. At large Reynolds numbers the inviscid equations of motion may be used to evaluate the integrand in (1.1) and thereby determine the variation of vorticity across the closed streamlines of an inviscid flow. Batchelor successfully applied this technique to several classes of flows with closed streamlines. Here we apply a similar technique to determine the vorticity distribution in an inviscid region of a class of flows which do not have closed streamlines.

The flow studied is constructed from a two-dimensional flow in a confined region by imposing an axial velocity normal to the plane of the two-dimensional motion, such that the velocity components are independent of the co-ordinate in the axial direction. Thus, the streamlines of the resultant motion are helical. An example typical of the flows to be considered is fully developed flow in a circular pipe which is spinning about its longitudinal axis. As the velocity is independent of the axial co-ordinate the axial pressure gradient is constant, and the two-dimensional flow in the cross-section (in this example, rigid body rotation) is independent of the axial velocity.

For flows where the streamlines are not closed (1.1) is inapplicable. However it is still possible to integrate the equations of motion along a streamline to obtain a relationship between the pressure and viscous forces. As the flow in the cross-section is independent of the axial velocity, the integral relationship splits into two independent

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integral conditions: one which represents properties of the axial velocity; and the other is the same as that used by Batchelor for two-dimensional flows with closed streamlines. By combining these exact results with the approximate inviscid flow equations we can determine the vorticity distribution in a helical flow at large Reynolds number.

2. Derivation of integral conditions

Consider a steady two-dimensional flow in a confined region, the motion being generated by the boundaries. A stream function ψ can be introduced, and in the plane of the motion a useful co-ordinate system is the orthogonal curvilinear co-ordinates (ψ, ξ), the lines ξ = constant being everywhere normal to the closed lines ψ = constant. The axis normal to the ψ, ξ plane is the z axis and, by definition, the velocity components are independent of z. Now impose a velocity in the z direction such that the velocity components in the plane of the previously existing two-dimensional flow are unaltered. This is achieved if the axial velocity is independent of z and the pressure p has the form

$$\frac{1}{\rho}p(\psi,\xi,z) = \frac{1}{\rho}p^*(\psi,\xi) - Gz, \tag{2.1}$$

where G is a constant and ρ is the density of the fluid. The condition on the axial velocity comes from continuity requirements, and the equations of motion dictate the form of the pressure.

The equation of steady motion for a viscous, incompressible fluid is

$$\mathbf{v}\times\boldsymbol{\omega} = \nabla H + \nu\,\mathrm{curl}\,\boldsymbol{\omega}, \tag{2.2}$$

where $\boldsymbol{\omega}$ is the vorticity, and is the curl of the velocity field $\mathbf{v}$, and ν is the kinematic viscosity of the fluid. The quantity H is the total head and is given by

$$H = \frac{1}{\rho}p + \tfrac{1}{2}\mathbf{v}\,.\,\mathbf{v}.$$

In the (ψ, ξ, z) co-ordinate system the velocity vector $\mathbf{v}$ is $[0, q, w]$ and the infinitesimal line element is $[d\psi/q,\ h_2\,d\xi,\ dz]$, where h_2 is an unknown function of ψ and ξ. The vorticity is then

$$\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3] = \left[\frac{1}{h_2}\frac{\partial w}{\partial \xi},\ -q\frac{\partial w}{\partial \psi},\ \frac{q}{h_2}\frac{\partial}{\partial \psi}(h_2 q)\right]. \tag{2.3}$$

When the viscous forces are negligible (2.2) may be replaced by the statement that H is constant on streamlines. From (2.1) the total head can be written

$$H = \frac{1}{\rho}p^*(\psi,\xi) + \tfrac{1}{2}q^2 - Gz + \tfrac{1}{2}w^2. \tag{2.4}$$

When G and w are both zero the flow is two-dimensional and H becomes

$$H^* = (1/\rho)p^* + \tfrac{1}{2}q^2 = H^*(\psi)$$

in the region of small viscous forces. By definition, G and w do not alter the flow described by ψ, so that in the region of small viscous forces (2.4) becomes

$$H = H^*(\psi) - Gz + \tfrac{1}{2}w^2, \tag{2.5}$$

which must be constant on streamlines.

The inviscid equations give no information as to the variation of H across the streamlines. To obtain such information we follow Batchelor (1956) and derive an exact result by integrating (2.2) along a streamline. The end points A and B of the path of integration are chosen so that one circuit of the projection of the streamline in the z-plane is made. We obtain

$$\int_{AB} \nabla H . d\mathbf{s} + \nu \int_{AB} \operatorname{curl} \boldsymbol{\omega} . d\mathbf{s} = 0, \tag{2.6}$$

where $d\mathbf{s}$ is the line element in the streamline.

The first integral in (2.6) is independent of the path of integration, and so may be taken along the straight line AB, which, from the definition of the endpoints, is in the direction of the z axis. Thus only $\partial H/\partial z$ will contribute to this term. The second integral may be expanded by expressing $d\mathbf{s}$ in terms of the (ψ, ξ, z) co-ordinate system, where the streamline element is $[0, h_2 d\xi, dz]$ and the equation of the streamline is

$$\frac{d\psi/q}{0} = \frac{h_2 d\xi}{q} = \frac{dz}{w}. \tag{2.7}$$

Thus (2.6) becomes

$$-G\int_A^B dz + \nu \int_A^B \frac{q}{h_2}\left(\frac{\partial}{\partial \psi}(\omega_2 h_2) - \frac{\partial}{\partial \xi}\left(\frac{\omega_1}{q}\right)\right) dz + \nu \oint \left(-q\frac{\partial \omega_3}{\partial \psi}\right) h_2 d\xi = 0, \tag{2.8}$$

where the components of the vorticity ω_1, ω_2 and ω_3 are given by (2.3).

The path of integration for the last term in (2.8) is the projection of the streamline onto the plane z = constant, i.e. the closed curve ψ = constant, and clearly this term is independent of both G and w. Thus (2.8) gives the two independent relationships

$$-\nu \oint q \frac{\partial \omega_3}{\partial \psi} h_2 d\xi = 0, \tag{2.9}$$

and

$$-G\int_A^B dz + \nu \int_A^B \frac{q}{h_2}\left\{\frac{\partial}{\partial \psi}(\omega_2 h_2) - \frac{\partial}{\partial \xi}\left(\frac{\omega_1}{q}\right)\right\} dz = 0. \tag{2.10}$$

Both (2.9) and (2.10) are exact results. Equation (2.9) was derived by Batchelor (1956) for two-dimensional flow with closed streamlines, and, when the path of integration lies wholly in the region of small viscous forces, the inviscid flow approximation $\omega_3 = \omega_3(\psi)$ may be used with (2.9) to derive the result that $\omega_3(\psi)$ is a constant, say ζ. We are interested in the conditions imposed by (2.10) on G and w as $\nu \to 0$, when the streamline path of integration lies wholly in the region of small viscous forces. Before turning away from the flow in the cross-section we note the result, given by (2.3):

$$\omega_3 = \frac{q}{h_2}\frac{\partial}{\partial \psi}(h_2 q) = \zeta, \quad \text{a constant.} \tag{2.11}$$

A more useful form of (2.10) can be obtained by using (2.7) to replace the z integrals by integrals around the closed contour ψ = constant. The balance expressed by (2.10) is then

$$-G\oint \frac{wh_2}{q}\,d\xi + \nu\oint w\left(\frac{\partial}{\partial\psi}(\omega_2 h_2) - \frac{\partial}{\partial\xi}\left(\frac{\omega_1}{q}\right)\right)d\xi = 0, \tag{2.12}$$

where the path of integration is in the effectively inviscid flow. By definition, velocity gradients remain $O(1)$ in the region of small viscous forces as $\nu \to 0$, and thus we can discard the possibility that large velocity gradients are responsible for the balance of terms in (2.12). Therefore, we have the ordering

$$G \sim O(\nu w/L^2) \quad \text{as} \quad \nu \to 0, \tag{2.13}$$

where L is a typical length scale. Using (2.13) in (2.5) and letting $\nu \to 0$ we obtain

$$H = H^*(\psi) + \tfrac{1}{2}w^2(\psi, \xi), \tag{2.14}$$

and for H to be constant on streamlines in the inviscid region w must be a function of ψ only, $w(\psi)$. The neglect of the pressure term, $-Gz$, in going from (2.5) to (2.14) is consistent with the inviscid result that H is constant on streamlines. Essentially the inviscid approximation is an energy equation obtained by neglecting the $O(\nu)$ viscous terms in the momentum equation as $\nu \to 0$. Here G is $O(\nu)$ compared to other terms in (2.5) and so may be neglected.

Substituting $w = w(\psi)$ in (2.3) and using this to simplify (2.12) we obtain

$$w(\psi)\left[G\oint \frac{h_2}{q}\,d\xi + \nu\oint \frac{\partial}{\partial\psi}\left(h_2 q \frac{dw}{d\psi}\right)d\xi\right] = 0. \tag{2.15}$$

The non-trivial solution of (2.15) is

$$w(\psi) = -G\psi/\nu\zeta + C\int^{\psi} \frac{d\psi}{\oint h_2 q\,d\xi} + D, \tag{2.16}$$

where C and D are arbitrary constants and (2.11) has been used to simplify the first term.

3. Discussion

The above result for the axial velocity distribution (and the earlier result (2.11) for the flow in the cross-section) has been derived assuming the streamline path of integration was wholly in the region of 'small viscous forces'. The definition of the regions of 'small viscous forces' requires some clarification. Batchelor (1956) defines these regions as parts of the flow where the viscous forces, suitably non-dimensionalized, are small compared with unity, and concludes that this is usually equivalent to the statement that 'viscous forces are small compared with pressure forces'. As the integral condition (2.10) requires pressure and viscous force to be the same magnitude, we see that this latter description of an effectively inviscid flow is inappropriate. Thus 'small viscous forces' must be interpreted in terms of the first of Batchelor's descriptions.

For the case $G \equiv 0$, the equation for w is equivalent to the equation for temperature in two-dimensional heat transfer. When the motion has uniform vorticity, then the inviscid approximation for the temperature is well known (see Burggraf 1966) and agrees with (2.16) with $G = 0$. In the heat transfer context, non-zero G may be interpreted as a uniform distribution of heat sources within the fluid, and in this case the inviscid approximation for the temperature distribution does not appear to have been previously published. Thus, as well as giving the axial velocity distribution in an inviscid helical flow, (2.16) also gives the inviscid approximation to the temperature field in a two-dimensional flow with uniform distribution of heat sources.

The unknown constants C and D in (2.16) will in general be found by examining the flow surrounding the inviscid region; matching the two regions should then determine C and D. Apart from matching conditions, there is also the condition that w must be regular which can be used to find the unknown constants. In particular, in any inviscid region where ψ has a local extremum C must be zero for w to be regular. This is readily shown by noting that, in a flow with uniform vorticity, the circulation $\oint h_2 q\, d\xi$ along a line $\psi =$ constant is $O(\psi - \psi_m)$ as $\psi \to \psi_m$, the local extremum. Thus the integral in (2.16) is logarithmically singular at $\psi = \psi_m$.

We conclude by applying (2.16) to the simple helical flow resulting from the fully developed motion of a viscous fluid inside an infinitely long circular cylinder. In cylindrical polar co-ordinates (r, Θ, z) the radial velocity is $-(1/r)\ (\partial\psi/\partial\Theta)$ and the azimuthal velocity is $\partial\psi/\partial r$, where ψ is the streamfunction of § 2. Thus the axial component of vorticity is

$$\omega_3 = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\Theta^2}\right)\psi. \tag{3.1}$$

The axial velocity w is generated by a constant pressure gradient, $\partial p/\partial z = -G$, and on the surface of the cylinder we impose the conditions

$$\left.\begin{aligned} \psi &= 0, \\ \partial\psi/\partial r &= \Omega a(1+\epsilon\sin\theta), \\ \text{and} \qquad w &= 0, \end{aligned}\right\} \quad \text{on} \quad r = a, \tag{3.2}$$

where the constant Ω has the units of angular velocity.

For non-zero ϵ Burggraf (1966) has shown that as $\nu \to 0$ (a, Ω fixed), the flow in the cross-section develops an inviscid core with uniform vorticity surrounded by a viscous boundary layer attached to the cylinder. As usual, the boundary-layer thickness is $O(\nu^{\frac{1}{2}})$ and in the layer $\partial/\partial r$ is $O(\nu^{-\frac{1}{2}})$. The streamfunction in the core is

$$\psi_1 = -\tfrac{1}{2}\zeta(a^2 - r^2), \tag{3.3}$$

where the (constant) axial vorticity ω_3 is ζ, which is given by (Wood 1957)

$$\zeta = \Omega(1 + \tfrac{1}{2}\epsilon^2)^{\frac{1}{2}}.$$

Direct application of (2.16) shows that the inviscid approximation for the axial velocity is

$$w_1 = -G\psi_1/\nu\zeta + D,$$

where C has been set to zero to make w_1 regular. As the shear stress on $r = a$ must balance the axial pressure gradient, we find that $w \sim O(\nu^{\frac{1}{2}})$ in the boundary layer. Thus, as $\psi_1 \to 0$ as $r \to a$, matching the core and boundary layer leads to the result that $D \sim O(\nu^{\frac{1}{2}})$, and hence the leading-order axial velocity distribution is

$$w_1 = -\frac{G\psi_1}{\nu\zeta} = \frac{G}{2\nu}(a^2 - r^2), \tag{3.4}$$

which is just Hagen–Poiseuille flow. This simple axial velocity distribution is due to the simple form of the inviscid motion in the cross-section. A more complicated example, where (2.16) greatly simplifies the determination of the axial velocity, is given in Blennerhassett (1976). Finally we note that, when $\epsilon \equiv 0$, the exact solution of the governing equations is given by (3.3) and (3.4), and then ψ_1 and w_1 describe the helical flow in a spinning pipe.

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